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Chazy's second-degree Painlevé equations

Christopher M Cosgrove

School of Mathematics and Statistics, The University of Sydney, Sydney, NSW 2006, Australia

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Abstract
We examine two sets of second-degree Painlevé equations derived by Chazy in 1909, denoted by systems (II) and (III). The last member of each set is a second-degree version of the Painlevé-VI equation, and there are no other second-order second-degree Painlevé equations in the polynomial class with this property. We map the last member of system (II) into the Fokas–Yortsos equation and demonstrate how both Schlesinger and Okamoto transformations for Painlevé-VI can be read off the Chazy equation. The 24 fundamental Schlesinger transformations were known to Garnier in 1943 while the 64 Okamoto transformations date from 1987. In an appendix, we gather together the solutions of the five members of system (II). System (III) is better known, being equivalent to Jimbo and Miwa's equations for the logarithmic derivatives of the tau functions of the six Painlevé transcendents. The last member, known to Painlevé in 1906, was written in a manifestly symmetric form by Jimbo and Miwa, suggesting many induced symmetries for Painlevé-VI. In particular, Schlesinger and Okamoto transformations for Painlevé-VI can be read off immediately.

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1. Introduction

In a short paper, Chazy (1909) tantalized his readers with two intriguing sets of second-degree differential equations. He began by writing down the six classical Painlevé equations, $P_{\rm I}, \ldots, P_{\rm VI}$, exactly as they had appeared in Painlevé (1906), which is the first time that they had been gathered together into a list. Chazy denoted them by system (I). With variables renamed but Painlevé's original parameters retained, system (I) is

$$w'' = 6w^2 + t, \tag{1.1}$$

$$w'' = 2w^3 + tw + \alpha, \tag{1.2}$$

$$w'' = \frac{1}{w}(w')^2 - \frac{1}{t}w' + \gamma w^3 + \frac{\alpha}{t}w^2 + \frac{\beta}{t} + \frac{\delta}{w}, \tag{1.3}$$

$$w'' = \frac{1}{2w}(w')^2 + \frac{3}{2}w^3 + 4tw^2 + 2(t^2 - \alpha)w + \frac{\beta}{w}, \tag{1.4}$$

$$w'' = \left\{\frac{1}{2w} + \frac{1}{w-1}\right\}(w')^2 - \frac{1}{t}w' + \frac{(w-1)^2}{t^2}\left\{\alpha w + \frac{\beta}{w}\right\} + \frac{\gamma w}{t} + \frac{\delta w(w+1)}{w-1}, \tag{1.5}$$

$$\begin{aligned} w'' = {} & \frac{1}{2}\left\{\frac{1}{w} + \frac{1}{w-1} + \frac{1}{w-t}\right\}(w')^2 - \left\{\frac{1}{t} + \frac{1}{t-1} + \frac{1}{w-t}\right\}w' \\ & + \frac{w(w-1)(w-t)}{t^2(t-1)^2}\left\{\alpha + \beta\frac{t}{w^2} + \gamma\frac{t-1}{(w-1)^2} + \delta\frac{t(t-1)}{(w-t)^2}\right\}. \end{aligned} \tag{1.6}$$

Unless otherwise stated, a prime will denote $\mathrm{d}/\mathrm{d}t$ throughout this paper. In general, we will use Leibniz' notation for differentiation with respect to other variables.

The first known appearance of each of these equations is as follows: P_{I} and P_{II} appear in Painlevé (1898a); P_{III} in Painlevé (1898b); P_{IV} in Gambier (1906a, 1906b); P_{V} in Gambier (1906c) together with an alternative derivation of P_{VI}; and P_{VI} in Fuchs (1905). The Fuchs paper is cited by both Gambier (1906c) and Painlevé (1906). An elementary special case of P_{V} was known to Painlevé (1902a). An elementary special case of P_{VI} was known to Picard (1889) and Painlevé (1893).

The discovery of P_{VI} by Fuchs was the first indication that Painlevé's own classification of second-order first-degree differential equations (Painlevé 1902a) was not complete. Gambier (1906a, 1906b, 1906c, 1907a, 1907b, 1910) reopened the investigation and completed the classification. He produced a list of 50 equations, the first 20 of which were in Painlevé (1902a). Gambier's list, with minor permutations and gauge changes, is the well-known list in Ince's classic textbook (Ince 1926). Gambier placed P_{VI} second-last in 49th position, and saved the last position for a beautiful equation containing three P_{I} functions in its coefficients. Ince permuted Gambier's last three equations, placing P_{VI} and the latter equation in positions L and XLVIII, respectively, this being a more natural ordering from the point of view of Painlevé classification.

For later convenience, we introduce Jimbo and Miwa's notation for the P_{VI} coefficient parameters (Jimbo and Miwa 1981, Jimbo 1982):

$$\alpha = \tfrac{1}{2}(\theta_\infty - 1)^2, \qquad \beta = -\tfrac{1}{2}\theta_0^2, \qquad \gamma = \tfrac{1}{2}\theta_1^2, \qquad \delta = \tfrac{1}{2}\left(1 - \theta_t^2\right). \tag{1.7}$$

Also, we will often speak of equations being equivalent to each other under some unspecified gauge transformation. The standard group of gauge transformations acting on Painlevé-type ordinary differential equations is the group of Möbius transformations,

$$\bar{w} = \frac{a(t)w + b(t)}{c(t)w + d(t)}, \qquad \bar{t} = \phi(t), \tag{1.8}$$

where t is the independent variable, w is the dependent variable, $ad - bc \neq 0$ and $\phi(t)$ is not constant. Because the second-degree equations appearing in this paper are all in the polynomial class, the gauge transformations acting on them consist of the subgroup of linear transformations with $c = 0$ and $d = 1$. As was well known to the earliest authors, the P_{VI} equation is invariant under a discrete group of 24 Möbius transformations of the form (1.8) which permute the distinguished values $w = \infty, 0, 1$ and t. When working with P_{VI}, it is useful to have a complete table on hand. To save space, we split it into two subgroups. The first consists of the identity and the following three involutions having $\bar{t} = t$:

$$\bar{w} = \frac{t}{w}, \qquad \begin{cases} \bar{\theta}_\infty = \theta_0 + 1, & \bar{\theta}_0 = \theta_\infty - 1, \\ \bar{\theta}_1 = \theta_t, & \bar{\theta}_t = \theta_1, \end{cases} \tag{1.9}$$

$$\bar{w} = \frac{w-t}{w-1}, \qquad \begin{cases} \bar{\theta}_\infty = \theta_1 + 1, & \bar{\theta}_0 = \theta_t, \\ \bar{\theta}_1 = \theta_\infty - 1, & \bar{\theta}_t = \theta_0, \end{cases} \tag{1.10}$$

$$\bar{w} = \frac{t(w-1)}{w-t}, \qquad \begin{cases} \bar{\theta}_\infty = \theta_t + 1, & \bar{\theta}_0 = \theta_1, \\ \bar{\theta}_1 = \theta_0, & \bar{\theta}_t = \theta_\infty - 1. \end{cases} \tag{1.11}$$

The second subgroup consists of the identity and the following five transformations having $\bar{\theta}_\infty = \theta_\infty$:

$$\bar{w} = 1 - w, \qquad \bar{t} = 1 - t, \qquad \bar{\theta}_0 = \theta_1, \qquad \bar{\theta}_1 = \theta_0, \qquad \bar{\theta}_t = \theta_t, \tag{1.12}$$

$$\bar{w} = w/t, \qquad \bar{t} = 1/t, \qquad \bar{\theta}_0 = \theta_0, \qquad \bar{\theta}_1 = \theta_t, \qquad \bar{\theta}_t = \theta_1, \tag{1.13}$$

$$\bar{w} = \frac{w-t}{1-t}, \qquad \bar{t} = \frac{t}{t-1}, \qquad \bar{\theta}_0 = \theta_t, \qquad \bar{\theta}_1 = \theta_1, \qquad \bar{\theta}_t = \theta_0, \tag{1.14}$$

$$\bar{w} = \frac{1-w}{1-t}, \qquad \bar{t} = \frac{1}{1-t}, \qquad \bar{\theta}_0 = \theta_1, \qquad \bar{\theta}_1 = \theta_t, \qquad \bar{\theta}_t = \theta_0, \tag{1.15}$$

$$\bar{w} = \frac{t-w}{t}, \qquad \bar{t} = \frac{t-1}{t}, \qquad \bar{\theta}_0 = \theta_t, \qquad \bar{\theta}_1 = \theta_0, \qquad \bar{\theta}_t = \theta_1. \tag{1.16}$$

In this paper, we do not consider quadratic or other algebraic transformations in w that are only admitted by P_{VI} with suitably restricted parameters.

The first of the aforementioned systems of second-degree Chazy equations is his system (II):

$$\left(\frac{d^2v}{dz^2} - 6v^2 - \alpha_1\right)^2 = z^2\left\{\left(\frac{dv}{dz}\right)^2 - 4v^3 - 2\alpha_1 v - \beta_1\right\}, \tag{1.17}$$

$$\left(\frac{d^2v}{dz^2} - 2v^3 - \alpha_1 v - \beta_1\right)^2 = -4(v - e^z)^2\left\{\left(\frac{dv}{dz}\right)^2 - v^4 - \alpha_1 v^2 - 2\beta_1 v - \gamma_1\right\}, \tag{1.18}$$

$$\left(\frac{d^2v}{dz^2} - \alpha_1 v - \beta_1\right)^2 = \frac{4v^2}{z^2}\left\{\left(\frac{dv}{dz}\right)^2 - \alpha_1 v^2 - 2\beta_1 v - \gamma_1\right\}, \tag{1.19}$$

$$\left(\frac{d^2v}{dz^2} - 6v^2 - \alpha_1 v - \beta_1\right)^2 = \left(\frac{2v}{z} - z\right)^2\left\{\left(\frac{dv}{dz}\right)^2 - 4v^3 - \alpha_1 v^2 - 2\beta_1 v - \gamma_1\right\}, \tag{1.20}$$

$$\left(\frac{d^2v}{dz^2} - 2v^3 - \alpha_1 v - \beta_1\right)^2 = 4\tan^2 z\left(v - \frac{\delta_1}{\sin z}\right)^2\left\{\left(\frac{dv}{dz}\right)^2 - v^4 - \alpha_1 v^2 - 2\beta_1 v - \gamma_1\right\}. \tag{1.21}$$

As before, we have renamed variables, but the parameters are the same as in Chazy (1909) except that we have placed a subscript 1 on each. These equations were later derived from first principles in a major second-degree Painlevé classification problem by Bureau (1972).

The second system of second-degree Chazy equations arose as first integrals of a set of third-order Painlevé-type equations that Chazy (1907) had derived two years earlier. This is Chazy's system (III):

$$\left(\frac{d^2u}{dx^2}\right)^2 + 4\left(\frac{du}{dx}\right)^3 + 2\left(x\frac{du}{dx} - u\right) = 0, \tag{1.22}$$

$$\left(\frac{\mathrm{d}^2u}{\mathrm{d}x^2}\right)^2+4\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^3+x\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2-u\frac{\mathrm{d}u}{\mathrm{d}x}+\alpha_2=0, \tag{1.23}$$

$$\left(\frac{\mathrm{d}^2u}{\mathrm{d}x^2}\right)^2+4\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^3+\left(x\frac{\mathrm{d}u}{\mathrm{d}x}-u\right)^2+\alpha_2\frac{\mathrm{d}u}{\mathrm{d}x}+\beta_2=0, \tag{1.24}$$

$$\begin{aligned}\left(\frac{\mathrm{d}^2u}{\mathrm{d}x^2}-\frac{2u}{x^2}\right)^2&+4\left(\frac{\mathrm{d}u}{\mathrm{d}x}-\frac{2u}{x}\right)\left(\frac{\mathrm{d}u}{\mathrm{d}x}+\frac{u}{x}\right)^2+\alpha_2x^2\left(x\frac{\mathrm{d}u}{\mathrm{d}x}-2u\right)^2+\beta_2x\left(x\frac{\mathrm{d}u}{\mathrm{d}x}-2u\right)\\&+\frac{\gamma_2}{x^2}\left(x\frac{\mathrm{d}u}{\mathrm{d}x}+u\right)+\delta_2=0,\end{aligned} \tag{1.25}$$

$$\begin{aligned}\left(\frac{\mathrm{d}^2u}{\mathrm{d}x^2}-2\wp(x)u\right)^2&+4\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^3-12\wp(x)u^2\frac{\mathrm{d}u}{\mathrm{d}x}+4\wp'(x)u^3\\&+\alpha_2\left\{\wp(x)\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2-\wp'(x)u\frac{\mathrm{d}u}{\mathrm{d}x}+\wp^2(x)u^2\right\}\\&+H(x)\frac{\mathrm{d}u}{\mathrm{d}x}-H'(x)u+\delta_2=0,\end{aligned} \tag{1.26}$$

where, in the last equation, $\wp(x)$ is the Weierstrass elliptic function $\wp(x;0,g_3)$ having $g_2=0$. The primes on $\wp$ and H denote $\mathrm{d}/\mathrm{d}x$. The parameter g_3 is not essential and can be normalized to any particular nonzero constant using scaling freedom in x and u, Chazy's choice being $g_3=1$. The function $H(x)$ is a Lamé function satisfying

$$H''(x)-2\wp(x;0,g_3)H(x)=0. \tag{1.27}$$

Chazy gave the explicit formula,

$$H(x)=\beta_2\frac{\sigma(x+h)}{\sigma(x)}\,\mathrm{e}^{-x\zeta(h)}+\gamma_2\frac{\sigma(x-h)}{\sigma(x)}\,\mathrm{e}^{x\zeta(h)}, \tag{1.28}$$

where h and $-h$ are the two unique roots of the equation $\wp(h;0,g_3)=0$ in a period parallelogram centred on the origin and $\sigma(x)$ and $\zeta(x)$ are Weierstrass sigma and zeta functions corresponding to $\wp(x;0,g_3)$.

Chazy announced that he had derived these ten equations by applying algebraic transformations to the six classical Painlevé equations, but he did not show the reader these transformations. More recently, Muğan and Sakka (1997, 1999) and Sakka and Muğan (1998) have taken up this idea and generated many second-degree Painlevé equations, mostly rational in the dependent variable. The five members of system (II) map, respectively, to $\mathrm{P_{IV}}$, $\mathrm{P_V}$, $\mathrm{P_{III}}$, $\mathrm{P_V}$ and $\mathrm{P_{VI}}$, and the five members of system (III) map, respectively, to $\mathrm{P_I}$, $\mathrm{P_{II}}$, $\mathrm{P_{IV}}$, $\mathrm{P_{III}}/\mathrm{P_V}$ and $\mathrm{P_{VI}}$. System (III) occurs naturally in a particular Painlevé classification problem (Cosgrove and Scoufis 1993) and can be embraced under a single master Painlevé equation. This system also arises in representations of the Painlevé transcendents as ratios of entire functions or at least analytic functions having no movable singularities (Jimbo and Miwa 1981).

Without knowing the precise problem that Chazy was investigating when he generated system (II), we cannot be sure that system (II) comprises the complete solution of that problem. However, it can be shown that system (II) is complete under the following hypotheses: (a) the background class of differential equations is second-order second-degree equations for $v(z)$ that are polynomial in v and $\mathrm{d}v/\mathrm{d}z$ and have the Painlevé property, (b) the square-free part on the right-hand side is quadratic in $\mathrm{d}v/\mathrm{d}z$, (c) that part has constant coefficients, and (d) the solution involves a Painlevé transcendent.

Painlevé (1902a) found equation (1.22) while expressing the $\mathrm{P_I}$ transcendent as a ratio of entire functions. Equations (1.23) and (1.24) can be extracted his corresponding analysis of P_{II} and P_{III}. Later, Painlevé (1906) described how he obtained a second-degree equation for a variable $\nu(x)$ by applying a transformation to P_{VI}, but there is an error in his equation (2). Chazy (1911) corrected Painlevé's result and produced a variable $t(x)$ that satisfies an equation equivalent to (1.26). Painlevé's equation (3), with $w(t)$ replacing his P_{VI} function $y(x)$, is

$$u(t) = \frac{t(t-1)(w')^2}{2w(w-1)(w-t)} - \frac{\alpha w}{t(t-1)} + \frac{\beta}{(t-1)w} + \frac{\gamma}{t(w-1)} - \frac{w'-\delta}{w-t},$$

w' denoting $\mathrm{d}w/\mathrm{d}t$, and Chazy's correction to Painlevé's equation (2) is

$$\nu(t) = u + \frac{\sqrt{2\alpha}w}{t(t-1)}.$$

Chazy's variable $t(x)$ is $\nu(x)/2$. The second-degree equation for $\nu(t)$ can be constructed by substituting the change of variable,

$$y = \tfrac{1}{4}\{2t(t-1)\nu + \alpha - \sqrt{2\alpha} - \beta - \gamma + \delta(2t-1)\},$$

into equation (3.13) for $y(t)$. However, the ν equation is not in the most elegant gauge.

Bureau (1964, 1972) attempted somewhat more ambitious Painlevé classification problems and ran into the same sets of second-degree equations. But, by then, the direct connection to the classical Painlevé equations was lost or forgotten, and Bureau was faced with the task of solving these equations from scratch. He was able to solve them all except (1.18), (1.20) and (1.21) (see also Bureau *et al* (1972)). Contact between system (II) and the Painlevé transcendents was reestablished by Fokas and Yortsos (1981), who found a second-degree equation gauge-equivalent to equation (1.21) and its explicit solution in terms of P_{VI}.

Many readers today would be familiar with some or all of these ten second-degree equations, except possibly for the fact that Chazy has written some of them in a less familiar gauge. Equations equivalent to members of systems (II) and (III) appear regularly, for example, in random matrix theory and related disciplines. System (III) is equivalent to the equations that Jimbo and Miwa (1981) gave for the logarithmic derivatives of the tau functions corresponding to the Painlevé transcendents, the last two members being in a different gauge. Special cases of equation (1.26) (in a gauge closer to equation (3.13)) have appeared in relativity applications (Cosgrove 1977, Ernst 1977, Dale 1978) where the authors at the time did not realize that the sixth Painlevé transcendent was involved. Schlesinger transformations are just below the surface of Ernst (1977) and can be lifted out with a small amount of work. Since members of system (III) are integrals of third- and fourth-order equations of the first degree, they appear in group-invariant reductions of several soliton equations.

Here, we are primarily interested in the fifth members (1.21) and (1.26) of each system because they map to P_{VI}. The other equations are limiting contractions of equations (1.21) and (1.26). Up to gauge, these are the only second-order second-degree Painlevé equations in the polynomial class that are solvable in terms of P_{VI}.

On the basis of the experience of mainly Russian authors in the preceding decades, Fokas and Ablowitz (1982) gathered together the known transformation properties of the Painlevé transcendents by relating them to second-degree equations. Ever since Gambier derived equations XXXIV, XXXV, XLV and XLVII (Ince numbering) in terms of P_{II} and XLII in terms of P_{IV}, it has been known that there is, in general, more than one way to carry out the reductions (because of $\pm$ signs in the formulae), and hence one can generate maps from P_{II} to itself and maps from P_{IV} to itself. With the aid of second-degree auxiliary equations, such maps become considerably more abundant, and comprehensive symmetry maps have been constructed for all of the Painlevé transcendents except the first. Interestingly, the equations that Fokas and Ablowitz selected for their demonstration were, up to a change of gauge, the first four equations of system (III) and the last equation of system (II). This meant that, at the time, P_{VI} did not quite fit the pattern of the other transcendents.

The four transformations (or one containing two $\pm$ signs) found by Fokas and Yortsos are Schlesinger transformations for P_{VI} which have been given a twist by the involution (1.9). Of course, if one particular Schlesinger transformation is known (either in a pure form or mixed with simpler transformations), the full set of 24 basic Schlesinger transformations ($\theta_\mu \to \theta_\mu \pm 1$ for exactly two of the four indices μ), can be generated easily by elementary operations. The natural setting is the matrix function $Y(x, t)$ satisfying the linear scattering problem (Jimbo and Miwa 1981, Jimbo 1982) rather than the P_{VI} function $w(t)$ itself, and complete results for 12 of the 24 Schlesinger transformations have been tabulated by Muğan and Sakka (1995a).

Pure Schlesinger transformations for P_{VI} were known to Garnier (1943), who saw an application to a theorem of Schwarz on minimal surfaces. Garnier did not write out a particular Schlesinger transformation in w explicitly, but gave a prescription for constructing 24 such transformations from Fuchs' associated linear problem (Fuchs 1905). He used the particular case given below by equations (2.6) and (2.7) as an illustration, but the reader needs to do a calculation to get the explicit map from P_{VI} to itself. With our variable $w(t)$ replacing Garnier's variable $\lambda(t)$, Garnier announced that his method would produce transformations of the form,

$$\bar{w} = \frac{M(w')^2 + Kw' + L}{H(w')^2 + Nw' + P}, \tag{1.29}$$

where the coefficients are polynomials in w and t. He compared his results to the relations of contiguity for hypergeometric functions. The effect on the theta parameters in the general case is given by

$$\bar{\theta}_{\mu_1} = \theta_{\mu_1} + \epsilon_{\mu_1}, \qquad \bar{\theta}_{\mu_2} = \theta_{\mu_2} + \epsilon_{\mu_2}, \qquad \bar{\theta}_{\mu_3} = \theta_{\mu_3}, \qquad \bar{\theta}_{\mu_4} = \theta_{\mu_4}, \tag{1.30}$$

where ϵ_{μ_1} and ϵ_{μ_2} denote ± 1 independently and $(\mu_1, \mu_2, \mu_3, \mu_4)$ is any permutation of the subscripts $(\infty, 0, 1, t)$ in equation (1.7).

Garnier devoted special attention to P_{VI} throughout his life and was aware that most results for P_{VI} had implications for some or all of the other five Painlevé transcendents. He found the asymptotics of P_{VI} near its critical points (Garnier 1916, 1917). He developed the theory of isomonodromic deformations of linear systems, originally applied to P_{VI} by Fuchs (1905), and generalized it to the Painlevé hierarchy now known as the Garnier system (Garnier 1912, 1917, 1919). He solved the Riemann–Hilbert problem for linear Fuchsian systems of differential equations and paid special attention to the associated monodromy problems for P_{VI} (Muğan and Sakka 1995b) and its hierarchy (Garnier 1926). He knew Fuchs' elementary solution of P_{VI} expressible in terms of hypergeometric functions (Fuchs 1907, Lukashevich and Yablonskii 1967a, 1967b) and generalized it to his hierarchy (Garnier 1912, 1917). (Elementary transcendental solutions of P_{II} and P_{IV} were known earlier to Painlevé (1902b) and Gambier (1910), respectively.) The first modern result on P_{VI} that goes substantially further than Garnier is Jimbo's derivation of the exact connection formulae for P_{VI} (Jimbo 1982). One can speculate that if Garnier (1943) had been better known, it is possible that the infrastructure of transformation properties and elementary solutions of the classical Painlevé equations would have been worked out decades earlier in a more systematic and unified way.

2. The Fokas–Yortsos equation

Let us take a closer look at Chazy's equation (1.21). Choose new variables t and $V(t)$ according to

$$\sin z = \frac{1+t}{1-t}, \qquad \cos z = \frac{2\mathrm{i}\sqrt{t}}{1-t}, \qquad v(z) = V(t) + \frac{\mu}{4}, \tag{2.1}$$

and new parameters κ, λ, μ and ν according to

$$\alpha_1 = 2\nu - \kappa^2 - \tfrac{1}{8}\mu^2, \qquad \beta_1 = \tfrac{1}{4}\kappa^2\mu,$$
$$\gamma_1 = \tfrac{1}{256}(16\nu + 4\kappa\mu - \mu^2)(16\nu - 4\kappa\mu - \mu^2), \qquad \delta_1 = \tfrac{1}{2}\lambda$$

(Sakka and Mugan 1998). Then equation (1.21) maps directly into the Fokas–Yortsos equation (Fokas and Yortsos 1981),

$$(L_1)^2 = (R_1)^2 S_1, \tag{2.2}$$

where

$$L_1 = V'' + \frac{3t-1}{2t(t-1)}V' + \frac{(4V+\mu)(2V^2+\mu V+2\nu) - 4\kappa^2 V}{4t(t-1)^2},$$
$$R_1 = \frac{(t+1)(4V+\mu) + 2\lambda(t-1)}{4t(t-1)},$$
$$S_1 = (V')^2 + \frac{(2V^2+\mu V+2\nu)^2 - 4\kappa^2 V^2}{4t(t-1)^2},$$

the prime denoting $\mathrm{d}/\mathrm{d}t$.

The solution of equation (2.2) in terms of P_{VI} is

$$V(t) = \frac{tw'}{w} + \frac{(\lambda-\kappa-1)w}{2(t-1)} + \frac{(\lambda+\kappa+1)t}{2(t-1)w} - \frac{\lambda(t+1)}{2(t-1)} - \frac{1}{2} - \frac{\mu}{4}, \tag{2.3}$$

where $w(t)$ satisfies the P_{VI} equation (1.6) with parameters

$$\alpha = \tfrac{1}{8}(\lambda-\kappa-1)^2, \qquad \beta = -\tfrac{1}{8}(\lambda+\kappa+1)^2,$$
$$\gamma = -\tfrac{1}{2}\nu + \tfrac{1}{32}(\mu-2\kappa)^2, \qquad \delta = \tfrac{1}{2}(\nu+1) - \tfrac{1}{32}(\mu+2\kappa)^2.$$

Fokas and Yortsos observed that their equation (2.2) is even in the parameter κ, whereas the expression for $V(t)$ in terms of P_{VI} is not. Thus there are two distinct reductions from equation (2.2) to P_{VI}. Because of the nonlinear parameter maps involving α and β, we get four distinct maps from P_{VI} to itself. These are the Fokas–Yortsos transformations. Their effect on the theta parameters is

$$\bar\theta_\infty = \theta_0 + \epsilon_0 + 1, \qquad \bar\theta_0 = \theta_\infty + \epsilon_\infty - 1, \qquad \bar\theta_1 = \theta_t, \qquad \bar\theta_t = \theta_1, \tag{2.4}$$

where ϵ_∞ and ϵ_0 denote ± 1 independently. The transformation formula for the P_{VI} function $w(t)$ is

$$\bar w = \frac{t(w-1)N_+N_- - (w-t)D_+D_-}{(w-1)N_+N_- - (w-t)D_+D_-}, \tag{2.5}$$

where

$$N_\pm = t(t-1)w' + \epsilon_0\theta_0(w-t) + \epsilon_\infty(\theta_\infty - 1)w(w-t) + (\pm\theta_t - 1)(t-1)w,$$
$$D_\pm = t(t-1)w' + \epsilon_0\theta_0 t(w-1) + \epsilon_\infty(\theta_\infty - 1)w(w-1) \pm \theta_1(t-1)w.$$

To untwist the Fokas–Yortsos transformation and get four pure Schlesinger transformations, apply the involution (1.9) to P_{VI} after the Fokas–Yortsos transformation. This involution acts very simply on the Fokas–Yortsos equation itself according to $\{V, \kappa, \lambda, \mu, \nu\} \to \{-V, \kappa, -\lambda, -\mu, \nu\}$. The effect of the composite transformation on P_{VI} is the pure Schlesinger transformation,

$$\bar{\theta}_\infty = \theta_\infty + \epsilon_\infty, \qquad \bar{\theta}_0 = \theta_0 + \epsilon_0, \qquad \bar{\theta}_1 = \theta_1, \qquad \bar{\theta}_t = \theta_t, \tag{2.6}$$

$$\bar{w} = \frac{t(w-1)N_+N_- - t(w-t)D_+D_-}{t(w-1)N_+N_- - (w-t)D_+D_-}. \tag{2.7}$$

The symmetry-generating power of the Fokas–Yortsos equation can be considerably enhanced by simply removing the additive constants $\mu/4$ and $-\mu/4$ from the right-hand sides of equations (2.1) and (2.3). Let us take this opportunity to rename two parameters,

$$\mu = 4\mu_1, \qquad \nu = \nu_1 + (\mu_1)^2.$$

Then the variable $V_1(t) := V(t) + \mu_1$ satisfies the second-degree equation,

$$(L_2)^2 = (R_2)^2 S_2, \tag{2.8}$$

where

$$L_2 = V_1'' + \frac{3t-1}{2t(t-1)}V_1' + \frac{2(V_1)^3 + (2\nu_1 - \kappa^2)V_1 + \kappa^2\mu_1}{t(t-1)^2},$$

$$R_2 = \frac{2(t+1)V_1 + \lambda(t-1)}{2t(t-1)},$$

$$S_2 = (V_1')^2 + \frac{((V_1)^2 + \nu_1)^2 - \kappa^2(V_1 - \mu_1)^2}{t(t-1)^2}.$$

This equation is invariant under the parameter maps,

$$\bar{\lambda} = \lambda, \qquad \bar{\mu}_1 = \frac{\kappa^2\mu_1}{\bar{\kappa}^2}, \qquad \bar{\nu}_1 = \nu_1 + \tfrac{1}{2}(\bar{\kappa}^2 - \kappa^2),$$

where $\bar{\kappa}$ satisfies the sextic equation,

$$(\bar{\kappa}^2 - \kappa^2)\left\{\bar{\kappa}^4 + (4\nu_1 - \kappa^2)\bar{\kappa}^2 + 4\kappa^2\mu_1^2\right\} = 0.$$

The root $\bar{\kappa} = -\kappa$ gives the four Fokas–Yortsos transformations above.

The four roots of the quartic factor yield the following 16 Okamoto transformations (Okamoto 1987). Let ϵ_∞, ϵ_0, ϵ_1 and ϵ_t denote ± 1 independently, and also independently of any previous usage. The theta parameters transform according to

$$\begin{aligned}
\bar{\theta}_\infty &= \tfrac{1}{2}(\epsilon_\infty\theta_\infty + \epsilon_0\theta_0 + \epsilon_1\theta_1 + \epsilon_t\theta_t + 1 - \epsilon_\infty),\\
\bar{\theta}_0 &= \tfrac{1}{2}(\epsilon_\infty\theta_\infty + \epsilon_0\theta_0 - \epsilon_1\theta_1 - \epsilon_t\theta_t + 1 - \epsilon_\infty),\\
\bar{\theta}_1 &= \tfrac{1}{2}(\epsilon_\infty\theta_\infty - \epsilon_0\theta_0 + \epsilon_1\theta_1 - \epsilon_t\theta_t + 1 - \epsilon_\infty),\\
\bar{\theta}_t &= \tfrac{1}{2}(\epsilon_\infty\theta_\infty - \epsilon_0\theta_0 - \epsilon_1\theta_1 + \epsilon_t\theta_t + 1 - \epsilon_\infty).
\end{aligned} \tag{2.9}$$

The P_{VI} function $w(t)$ transforms according to

$$\bar{w} = w + N/D, \tag{2.10}$$

where

$$N = (\epsilon_\infty\theta_\infty - \epsilon_0\theta_0 - \epsilon_1\theta_1 - \epsilon_t\theta_t + 1 - \epsilon_\infty)w(w-1)(w-t),$$

$$D = t(t-1)w' + \epsilon_0\theta_0(w-1)(w-t) + \epsilon_1\theta_1 w(w-t) + (\epsilon_t\theta_t - 1)w(w-1).$$

Forty-eight additional transformations of this type can be constructed by composition with the involutions (1.9)–(1.11). (Of course, we can also iterate with the transformations (1.12)–(1.16), but we do not get any new Okamoto transformations if we restrict attention to results with $\bar{t} = t$.)

The 64 Okamoto transformations, being of degree 1 in w', are simpler than the Schlesinger and Fokas–Yortsos transformations, which are of degree 2 in w'. The latter can each be factorized into two Okamoto transformations. Conversely, compositions of two Okamoto transformations can yield transformations of degree up to 4 in w'. Under Painlevé's contractions, the Okamoto transformations for P_{VI} carry down to the Gromak transformations for P_V (Gromak 1976a, 1976b) and the Gambier–Lukashevich transformations for P_{IV} (Gambier 1910, Lukashevich 1967a, 1967b, Bureau 1980).

In the appendix, we solve Chazy's equation (1.21) directly in terms of P_{VI} without using the Fokas–Yortsos equation as an intermediate step. It will be noted that the induced group of symmetries of P_{VI} that leaves equation (1.21) invariant is larger than the group that leaves equation (2.8) invariant because there are several distinct maps from equation (1.21) to (2.8).

3. Master Painlevé equations

The sixth Painlevé equation (1.6) is a 'master Painlevé equation' in the sense that particular limiting contractions of P_{VI} yield 25 of the 50 equations in Gambier's and Ince's lists (Gambier 1910, Ince 1926). These equations have Ince numbers I, II, III, IV (P_I), VII, VIII, IX (P_{II}), XI, XII, XIII (P_{III}), XVII with $m = 2$, XVIII, XIX, XX, XXIX, XXX, XXXI (P_{IV}), XXXII, XXXIII, XXXIV, XXXVII, XXXVIII, XXXIX (P_V), XLIX and L (P_{VI}). The contractions from P_{VI} to the other five Painlevé equations are well known (Painlevé 1906, Ince 1926). These can be supplemented by a contraction from P_{IV} to Ince-XXXIV (Kitaev 1992) and two different contractions of Ince-XXXIV yielding P_I and Ince-XX. The other 17 equations involve elliptic or simpler functions.

To get Ince-XLIX (Gambier's 48th equation), replace t by $a + \epsilon t$ in P_{VI} and suitably scale the parameters. The limiting equation as $\epsilon \to 0$ is

$$w'' = \frac{1}{2}\left\{\frac{1}{w} + \frac{1}{w-1} + \frac{1}{w-a}\right\}(w')^2 + w(w-1)(w-a)\left\{\alpha + \frac{\beta}{w^2} + \frac{\gamma}{(w-1)^2} + \frac{\delta}{(w-a)^2}\right\}. \tag{3.1}$$

By suitably generalizing the gauge in Ince-XLIX, we can capture 17 of the 25 contractions of P_{VI} by just taking particular values of the parameters. Let $P(w)$ and $Q(w)$ be arbitrary polynomials in w of degree at most 4 with constant coefficients, P being not identically zero. Then the equation,

$$w'' = \frac{P'(w)}{2P(w)}(w')^2 + \frac{P(w)Q'(w) - Q(w)P'(w)}{P(w)}, \tag{3.2}$$

with first integral, $(w')^2 = KP(w) + 2Q(w)$, has the Painlevé property. Only four of its ten parameters are essential. When $P(w)$ is either cubic or quartic with no square factors, equation (3.2) is equivalent to Ince-XLIX under a Möbius transformation (1.8) with constant coefficients. It is the standard form (3.1) of Ince-XLIX when $P(w) = w(w-1)(w-a)$. When $P(w)$ has square factors or is of lower degree, equation (3.2) separates into equations equivalent to the other 16 equations and includes each of their standard forms. In Painlevé classification problems, we call an equation like (3.2) a master Painlevé equation because it embraces several classification subcases at once (see Cosgrove (1997)). A different usage of the term applies to equations (3.4) and (3.14).

It is instructive to see how far we can rewrite P_{VI} in a general gauge. Let

$$P(t, w) = A(t)w^4 + B(t)w^3 + C(t)w^2 + D(t)w + E(t),$$

where the five coefficient functions are arbitrary except that $A(t)$ and $B(t)$ are not both zero and $P(t, w)$ has no square factors in w. Construct the relative invariants,

$$
\begin{aligned}
J_1(t) &= 12AE - 3BD + C^2, \\
J_2(t) &= 27AD^2 - 72ACE - 9BCD + 27B^2E + 2C^3, \\
J_3(t) &= \frac{4(J_1)^3 - (J_2)^2}{27}, \\
J_4(t) &= \frac{1}{27}\left(3J_2\frac{\mathrm{d}J_1}{\mathrm{d}t} - 2J_1\frac{\mathrm{d}J_2}{\mathrm{d}t}\right), \\
J_5(t) &= \frac{1}{J_4}\frac{\mathrm{d}J_4}{\mathrm{d}t} - \frac{1}{J_3}\frac{\mathrm{d}J_3}{\mathrm{d}t}.
\end{aligned}
$$

The invariant J_3 is A^6 times the discriminant of the quartic $P(t, w)$ when $A \neq 0$ and is B^6 times the discriminant of the cubic $P(t, w)$ when $A = 0$. Under the stated hypotheses, J_3 does not vanish. The nonvanishing of J_4 is a separate hypothesis, the case $J_4 = 0$ yielding Ince-XLIX in a general gauge instead of $\mathrm{P_{VI}}$. Under a gauge change of the form (1.8), $\mathrm{P_{VI}}$ can be transformed into the equation,

$$
w'' = \frac{P_w(t, w)}{2P(t, w)}(w')^2 + \left\{\frac{P_t(t, w)}{P(t, w)} + J_5(t)\right\} w' + \frac{R(t, w)}{P(t, w)}, \tag{3.3}
$$

where the subscripts t and w denote partial differentiation and $R(t, w)$ is a polynomial in w of degree 6, in general, but can be of lower degree in particular cases. Unfortunately, the coefficients of $R(t, w)$ cannot be expressed in terms of symmetric functions of the roots of $P(t, w)$, except in the elementary Picard case $\alpha = \beta = \gamma = 0$ and $\delta = 1/2$, are so are rather complicated. Also, because J_3 appears on the denominator, the reductions to $\mathrm{P_V}$ in a general gauge, and so on, where $P(t, w)$ has square factors, requires special handling. We see that equation (3.3) does not fulfil the role of a master Painlevé equation that crosses classification boundaries quite as well as equation (3.2) or some of the rational second-degree equations in Cosgrove (1997).

Let us now take a closer look at the fifth member of Chazy's system (III). This equation appears in the literature in two qualitatively different gauges, with internal variations within each. Chazy's original gauge, in which elliptic and Lamé functions appear in the coefficients, occurs naturally in the Painlevé classification of third-order equations (Chazy 1911, Bureau 1964, Cosgrove 2000).

Of the 13 canonical types of third-order equations appearing in Chazy (1911), the first is identified by the reduced equation,

$$
\frac{\mathrm{d}^3U}{\mathrm{d}x^3} = -6\left(\frac{\mathrm{d}U}{\mathrm{d}x}\right)^2,
$$

and corresponding full equation,

$$
\begin{aligned}
\frac{\mathrm{d}^3U}{\mathrm{d}x^3} = -6\left(\frac{\mathrm{d}U}{\mathrm{d}x}\right)^2 &+ A_1(x)\frac{\mathrm{d}^2U}{\mathrm{d}x^2} + B_1(x)U\frac{\mathrm{d}U}{\mathrm{d}x} + C_1(x)U^3 \\
&+ D_1(x)\frac{\mathrm{d}U}{\mathrm{d}x} + E_1(x)U^2 + F_1(x)U + G_1(x),
\end{aligned}
$$

whose coefficients are to be determined by running standard Painlevé tests. An admissible choice of gauge is $A_1(x) = 0$ and $E_1(x) = D_1(x)$. Then the compatibility conditions in the Laurent expansion about a movable simple pole (resonances 1 and 6) force $B_1(x) = C_1(x) = 0$ and supply three differential constraints on the remaining coefficients.

The final form of the Chazy-I equation is

$$\frac{d^3U}{dx^3} = 6\left\{-\left(\frac{dU}{dx}\right)^2 + A(x)\left(\frac{dU}{dx} + U^2\right) + B(x)U + C(x)\right\}, \tag{3.4}$$

where the coefficient functions $A(x)$, $B(x)$ and $C(x)$ satisfy

$$A'' = 6A^2, \qquad B'' = 6AB, \qquad C'' = B^2 + 2AC, \tag{3.5}$$

the primes on A, B and C denoting d/dx. This is the most compact of several alternative forms of the Chazy-I equation. It appears in this form in Chazy (1911) and in a closely related gauge in Chazy (1907). It contains all of system (III) and all of the Painlevé transcendents in their full generality after integration. Because of the latter property, we call it a 'master Painlevé equation', this being a different usage of the phrase to that above. Up to a translation in x, $A(x)$ is one of the following three functions:

$$\wp(x; 0, g_3), \qquad 1/x^2, \qquad 0.$$

As already mentioned, we could use some left over gauge freedom to normalize g_3. A translation in U could also be used to remove one of the constants in $B(x)$. Thus only three of the six parameters in equation (3.4) are essential.

The first case, where $A(x)$ is a Weierstrass elliptic function, integrates up to an equation equivalent to (1.26) and can be solved in terms of P_{VI}. The second case integrates up to an equation equivalent to (1.25) and can be solved in terms of P_V and/or P_{III}. The third case subdivides into equations that integrate up to (1.22)–(1.24) and can be solved in terms of P_I, P_{II} or P_{IV} functions, respectively.

A first integral of (3.4) is

$$\begin{aligned}\left(\frac{d^2U}{dx^2} - 2AU - B\right)^2 &= -4\left(\frac{dU}{dx}\right)^3 + 12AU^2\frac{dU}{dx} - 4A'U^3 + 4A\left(\frac{dU}{dx}\right)^2 \\ &\quad + (12B - 4A')U\frac{dU}{dx} + (4A^2 - 6B')U^2 + (12C - 2B')\frac{dU}{dx} \\ &\quad + (4AB - 12C')U + B^2 - 12\int BC\, dx + K.\end{aligned} \tag{3.6}$$

We can easily express the variables and coefficient functions in equation (1.26) in terms of those in equation (3.6) or vice versa when $A(x) = \wp(x; 0, g_3)$ with $g_3 \neq 0$. Two first integrals of the A and B equations (3.5) are

$$(A')^2 = 4A^3 - g_3, \qquad AB' - BA' = k_1.$$

The variables $u(x)$ and $U(x)$ are related by

$$u = U - U_0, \qquad U_0 = \frac{k_1A' - g_3B}{2g_3A}, \tag{3.7}$$

and U_0 satisfies $dU_0/dx = k_1A/g_3$ and $d^2U_0/dx^2 - 2AU_0 = B$. The Lamé function $H(x)$ satisfying (1.26) and the integral in (3.6) are given by

$$\begin{aligned}H &= \frac{3g_3(B^2 - 4AC) + 3k_1^2}{g_3A}, \\ \int BC\, dx &= \frac{1}{12g_3A^3}\{6A^2C'(g_3B - k_1A') + 12k_1A^4C \\ &\quad - A'B(g_3B^2 - 3k_1^2) + 3k_1(A^3 - g_3)B^2 + k_1^3\},\end{aligned}$$

where we selected a particular integration constant. Then the constants α_2 and δ_2 in equation (1.26) are given by

$$\alpha_2 = \frac{12k_1}{g_3} - 4, \qquad \delta_2 = \frac{k_1^2(5k_1 + g_3)}{g_3^2} - K.$$

The most useful form of equation (1.26) is one with the same independent variable t as the P_{VI} function $w(t)$. Suppose that $u(x)$ satisfies equation (1.26) and write $A(x) = \wp(x; 0, g_3)$ as above. Let μ be one of the roots of $\mu^6 = g_3/27$ and let

$$x = \frac{1}{3\mu}\int \frac{\mathrm{d}t}{t^{2/3}(t-1)^{2/3}}. \tag{3.8}$$

The inverse $t(x)$ is the elliptic function $\{1 + \wp'(\mu x; 0, -1)\}/2$. The elliptic function $A(x)$ is given by

$$A(x) = \frac{\mu^2(t^2 - t + 1)}{t^{2/3}(t-1)^{2/3}}, \tag{3.9}$$

$$A'(x) = \frac{\mu^3(t+1)(t-2)(2t-1)}{t(t-1)}. \tag{3.10}$$

The Lamé function $H(x)$ is given by

$$H(x) = \frac{D_1 t + D_2}{t^{1/3}(t-1)^{1/3}}, \tag{3.11}$$

where D_1 and D_2 are constants linearly related to β_2 and γ_2. The solution $u(x)$ of equation (1.26) is given by

$$u(x) = \frac{\mu(72y - \alpha_2(2t-1))}{24t^{1/3}(t-1)^{1/3}}, \tag{3.12}$$

where the variable $y(t)$ satisfies the second-degree equation,

$$t^2(t-1)^2(y'')^2 = -4y'(ty' - y)\{(t-1)y' - y\} + A_1(y')^2 + A_2(ty' - y) + A_3 y' + A_4. \tag{3.13}$$

This is precisely the equation denoted by SD-I.a in Cosgrove and Scoufis (1993). The coefficients are given in terms of α_2, δ_2, D_1 and D_2 by

$$A_1 = -\frac{\alpha_2}{6}, \qquad A_2 = -\frac{\mu^2 D_1}{3g_3}, \qquad A_3 = \frac{\alpha_2^2}{144} - \frac{\mu^2 D_2}{3g_3},$$
$$A_4 = \frac{\mu^2\alpha_2(D_1 + 2D_2)}{216g_3} - \frac{\delta_2}{27g_3} - \frac{\alpha_2^3}{11\,664}.$$

Equation SD-I.a is gauge-equivalent to the generic case of the ten-parameter equation denoted by SD-I:

$$\begin{aligned}(y'')^2 = -4\{c_1t^3 + c_2t^2 + c_3t + c_4\}^{-2}\{&c_1(ty' - y)^3 + c_2y'(ty' - y)^2 \\ &+ c_3(y')^2(ty' - y) + c_4(y')^3 + c_5(ty' - y)^2 + c_6y'(ty' - y) \\ &+ c_7(y')^2 + c_8(ty' - y) + c_9y' + c_{10}\}.\end{aligned} \tag{3.14}$$

(Differentiating out the parameter c_{10} produces a nine-parameter version of the Chazy-I equation (Cosgrove 2000).) The generic case occurs when the polynomial $c_1t^3 + c_2t^2 + c_3t + c_4$ is either cubic or quadratic with no square factors, in which case it can be normalized to $t(t-1)$ by a gauge transformation (Möbius in t, linear in y). This case involves P_{VI}. The generic and nongeneric cases together involve all six Painlevé transcendents in their full generality. Conversely, every case of equation SD-I can be solved with one of the six Painlevé transcendents or simpler functions. Thus SD-I qualifies unconditionally as a master Painlevé equation in the second sense.

To solve equation SD-I.a in terms of P_{VI} (Jimbo and Miwa 1981, Jimbo 1982), let

$$\begin{aligned}
A_1 &= \alpha - \beta + \gamma - \delta - \sqrt{2\alpha} + 1,\\
A_2 &= (\beta + \gamma)(\alpha + \delta - \sqrt{2\alpha}),\\
A_3 &= (\gamma - \beta)(\alpha - \delta - \sqrt{2\alpha} + 1) + \tfrac{1}{4}(\alpha - \beta - \gamma + \delta - \sqrt{2\alpha})^2,\\
A_4 &= \tfrac{1}{4}(\gamma - \beta)(\alpha + \delta - \sqrt{2\alpha})^2 + \tfrac{1}{4}(\beta + \gamma)^2(\alpha - \delta - \sqrt{2\alpha} + 1),
\end{aligned}$$

where $\sqrt{2\alpha}$ can take either sign. Then

$$\begin{aligned}
y = {} & \frac{t^2(t-1)^2}{4w(w-1)(w-t)}\left\{w' - \frac{w(w-1)}{t(t-1)}\right\}^2\\
& + \frac{1}{8}(1-\sqrt{2\alpha})^2(1-2w) - \frac{1}{4}\beta\left(1 - \frac{2t}{w}\right)\\
& - \frac{1}{4}\gamma\left(1 - \frac{2(t-1)}{w-1}\right) + \frac{1}{8}(1-2\delta)\left(1 - \frac{2t(w-1)}{w-t}\right),
\end{aligned} \tag{3.15}$$

$$y' = -\frac{t(t-1)}{4w(w-1)}\left\{w' - \sqrt{2\alpha}\frac{w(w-1)}{t(t-1)}\right\}^2 - \frac{1}{2}\beta\frac{w-t}{(t-1)w} - \frac{1}{2}\gamma\frac{w-t}{t(w-1)}, \tag{3.16}$$

where $w(t)$ is a solution of the P_{VI} equation (1.6). The inverse map is

$$w = -\frac{tS + 8(\sqrt{2\alpha}-1)t^2(t-1)^2y''}{R}, \tag{3.17}$$

where

$$\begin{aligned}
R = {} & 16(\sqrt{2\alpha}-1)^2 t(t-1)y' + \{4y - (\alpha - \sqrt{2\alpha} + \delta)(2t-1) + \beta + \gamma\}^2\\
& + 8(\sqrt{2\alpha}-1)^2(\beta t + \gamma t - \gamma),
\end{aligned} \tag{3.18}$$

$$\begin{aligned}
S = {} & 4(t-1)\big\{4y - (\alpha - \sqrt{2\alpha} + \delta)(2t-1) - 2(\sqrt{2\alpha}-1)^2 + \beta + \gamma\big\}y'\\
& - (4y - \alpha + \sqrt{2\alpha} - \delta)\{4y - (\alpha - \sqrt{2\alpha} + \delta)(2t-1)\}\\
& - (\beta + \gamma)\{8y + 2(3\alpha - 3\sqrt{2\alpha} - \delta + 2)t + \beta + \gamma\}\\
& - 4(\beta - \gamma)(\sqrt{2\alpha} - 1)^2.
\end{aligned} \tag{3.19}$$

Equation SD-I.a (3.13) was written in a manifestly symmetric form by Jimbo and Miwa (1981) and Jimbo (1982) which reveals immediately a large number of symmetries of the P_{VI} function, including the Schlesinger and Okamoto transformations (Okamoto 1987). Introduce Jimbo and Miwa's theta parameters given by equation (1.7) and specifically set $\sqrt{2\alpha} = 1 - \theta_\infty$. Then Jimbo and Miwa's presentation of equation SD-I.a (with different names for variables) is

$$t^2(t-1)^2y'(y'')^2 = -\{(2t-1)(y')^2 - 2yy' + M\}^2 + (y'+m_1)(y'+m_2)(y'+m_3)(y'+m_4), \tag{3.20}$$

where

$$\begin{aligned}
m_1 &= \tfrac{1}{4}(\theta_\infty + \theta_t)^2, & m_2 &= \tfrac{1}{4}(\theta_\infty - \theta_t)^2,\\
m_3 &= \tfrac{1}{4}(\theta_0 + \theta_1)^2, & m_4 &= \tfrac{1}{4}(\theta_0 - \theta_1)^2,\\
M &= \tfrac{1}{16}(\theta_\infty + \theta_t)(\theta_\infty - \theta_t)(\theta_0 + \theta_1)(\theta_0 - \theta_1).
\end{aligned}$$

Symmetries of the P_{VI} parameters that leave equation (3.20) invariant jump off the page at the reader. We can see obvious permutations of $\pm\theta_\infty \pm \theta_t$ and $\pm\theta_0 \pm \theta_1$ that permute the m_i and preserve the overall sign of M. It is a straightforward calculation to lift the parameter maps to maps from $w(t)$ to itself. By combining with the group of 24 Möbius transformations in w generated by (1.9)–(1.16), an infinite number of transformations can be generated, all of which are products of Okamoto and simpler transformations.

We distinguish three basic types of symmetries that leave SD-I.a invariant. First, sign changes $\theta_\mu \to -\theta_\mu$ are trivial and have no effect on P_{VI} for $\mu = 0, 1$ and t. Nevertheless, they are important in compositions with other transformations. The sign change $\theta_\infty \to -\theta_\infty$, on the other hand, generates a transformation of degree 4 in w' which has the same effect on P_{VI} as the Schlesinger transformations $\theta_\infty \to \theta_\infty \pm 2$. These can be factorized into two basic Schlesinger transformations of the form (2.7) or into two Fokas–Yortsos transformations or four Okamoto transformations.

Second, the four theta parameters can undergo two disjoint interchanges. This gives rise to 12 transformations for P_{VI} of degree 2 in w' which are of the same character as the Fokas–Yortsos transformations, and include the latter. As above, we prefer to untwist these transformations using the involutions (1.9)–(1.11). We then get all 24 of the basic Schlesinger transformations, whose effect on the theta parameters is given by equation (1.30).

Third, we get eight of the sixteen Okamoto transformations given above by equations (2.9) and (2.10). The eight are identified by $\epsilon_\infty = +1$. To get the other eight with $\epsilon_\infty = -1$, we just set $\sqrt{2\alpha} = \theta_\infty - 1$. The Fokas–Yortsos transformations are products of two Okamoto transformations.

Finally, changing the sign of $\bar\theta_\infty$ in (2.9) yields sixteen transformations (eight with $\sqrt{2\alpha} = 1 - \theta_\infty$ and eight with $\sqrt{2\alpha} = \theta_\infty - 1$) that factorize into three Okamoto transformations. All of the transformations mapping P_{VI} to itself that leave SD-I.a invariant can be factorized into one, two, three or four Okamoto transformations.

Appendix. Solutions of Chazy's system (II)

We gather together the solutions of the five second-degree equations of Chazy's system (II). A corresponding set of solutions of system (III) is readily available (Jimbo and Miwa 1981, Cosgrove and Scoufis 1993). For convenience as a reference, we include an optional scaling parameter A in the first four equations. The symbols ϵ_j for $j = 1, 2, \ldots$ each denote ± 1. The primes on w denote $\mathrm{d}/\mathrm{d}t$.

The first Chazy equation is

$$\left(\frac{\mathrm{d}^2v}{\mathrm{d}z^2} - 6v^2 - \alpha_1\right)^2 = 4A^2z^2\left\{\left(\frac{\mathrm{d}v}{\mathrm{d}z}\right)^2 - 4v^3 - 2\alpha_1 v - \beta_1\right\}. \tag{A.1}$$

Let q be any root of the cubic equation,

$$4q^3 + 2\alpha_1 q + \beta_1 = 0,$$

and let

$$\alpha = \frac{3q + 2\epsilon_1 A}{2A}, \qquad \beta = \frac{3q^2 + 2\alpha_1}{2A^2}.$$

Then the solution of equation (A.1) is

$$z = A^{-1/2}t, \qquad v(z) = \tfrac{1}{2}A(\epsilon_1 w' + w^2 + 2tw) - \tfrac{1}{2}q, \tag{A.2}$$

where $w(t)$ satisfies the P_{IV} equation (1.4) with parameters α and β.

The multiplicity of values of α and β implies several direct maps from P_{IV} to P_{IV}. These include the Gambier–Lukashevich and Schlesinger transformations. Similar comments apply to the remaining cases below.

The second Chazy equation is

$$\left(\frac{\mathrm{d}^2 v}{\mathrm{d}z^2} - 2v^3 - \alpha_1 v - \beta_1\right)^2 = -4(v - A\,\mathrm{e}^z)^2\left\{\left(\frac{\mathrm{d}v}{\mathrm{d}z}\right)^2 - v^4 - \alpha_1 v^2 - 2\beta_1 v - \gamma_1\right\}. \tag{A.3}$$

Let q_1 and q_2 be any two distinct roots of the quartic equation,

$$q^4 + \alpha_1 q^2 + 2\beta_1 q + \gamma_1 = 0,$$

and let

$$r = q_1 + q_2, \qquad \alpha = (r^3 + 2\alpha_1 r - 4\epsilon_1\beta_1)/(8r),$$
$$\beta = -(r^3 + 2\alpha_1 r + 4\epsilon_1\beta_1)/(8r), \qquad \gamma = 2A(\epsilon_1 r - i), \qquad \delta = 2A^2.$$

Then the solution of equation (A.3) is

$$z = \log t, \qquad v(z) = \frac{t(\mathrm{i}w' - 2Aw)}{(w-1)^2} - \frac{\epsilon_1 r(w+1)}{2(w-1)}, \tag{A.4}$$

where $w(t)$ satisfies the P_V equation (1.5) with the indicated parameters. (If $r = 0$ then $\beta_1 = 0$ and the above expressions hold except that α and $-\beta$ become the roots of the quadratic equation $4x^2 - 2\alpha_1 x + \gamma_1 = 0$.)

The third Chazy equation is

$$\left(\frac{\mathrm{d}^2 v}{\mathrm{d}z^2} - \alpha_1 v - \beta_1\right)^2 = \frac{4A^2 v^2}{z^2}\left\{\left(\frac{\mathrm{d}v}{\mathrm{d}z}\right)^2 - \alpha_1 v^2 - 2\beta_1 v - \gamma_1\right\}. \tag{A.5}$$

This is the primed version of equation SD-III in Cosgrove and Scoufis (1993). When α_1 and β_1 are not both zero, let q be any root of the quadratic equation (linear if $\alpha_1 = 0$),

$$\alpha_1 q^2 + 2\beta_1 q + \gamma_1 = 0,$$

and let

$$\gamma = \mu^2, \qquad \alpha = -\mu(2\epsilon_1 A q + 1),$$
$$\beta = \mu\{4\epsilon_1 A\beta_1 + \alpha_1(2\epsilon_1 A q + 1)\}, \qquad \delta = -\mu^2\alpha_1^2,$$

where μ is another optional scaling parameter. The solution of equation (A.5) when α_1 and β_1 are not both zero is

$$z = 2\mu t, \qquad v(z) = \frac{\epsilon_1 t}{2A}\left\{\frac{w' - \mu\alpha_1}{w} + \mu w\right\}, \tag{A.6}$$

where $w(t)$ satisfies the P_{III} equation (1.3) with the indicated parameters. When $\alpha_1 = 0$ and β_1 and γ_1 are not both zero, the solution of equation (A.5) is

$$z = t, \qquad v(z) = \frac{\epsilon_1 t}{2A}\left\{\frac{w'}{w} + \beta_1 w\right\}, \tag{A.7}$$

where $w(t)$ satisfies the P_{III} equation with parameters,

$$\gamma = \beta_1^2, \qquad \alpha = \epsilon_1 A\gamma_1 - \beta_1, \qquad \beta = \epsilon_1 A, \qquad \delta = 0.$$

On the overlap of the two cases, the solutions given differ by scalings.

The fourth Chazy equation is

$$\left(\frac{\mathrm{d}^2 v}{\mathrm{d}z^2} - 6v^2 - \alpha_1 v - \beta_1\right)^2 = \left(\frac{2Av}{z} - \frac{z}{A}\right)^2\left\{\left(\frac{\mathrm{d}v}{\mathrm{d}z}\right)^2 - 4v^3 - \alpha_1 v^2 - 2\beta_1 v - \gamma_1\right\}. \tag{A.8}$$

Let q be any root of the cubic equation,

$$4q^3 + \alpha_1 q^2 + 2\beta_1 q + \gamma_1 = 0,$$

and let

$$\alpha = -\frac{A^2\{(12q+\alpha_1)^2 + 4(24\beta_1 - \alpha_1^2)\}}{384}, \qquad \beta = -\frac{\{A(4q+\alpha_1) - 4\epsilon_1\}^2}{128},$$
$$\gamma = \frac{2Aq - \epsilon_1}{2A}, \qquad \delta = -\frac{1}{2A^2}.$$

The solution of equation (A.8) is

$$z = \sqrt{2t}, \qquad v(z) = \frac{t(\epsilon_1 A w' + w)}{A^2 w(w-1)} - \frac{4q+\alpha_1}{8} + \frac{A(4q+\alpha_1) - 4\epsilon_1}{8Aw}, \tag{A.9}$$

where $w(t)$ satisfies the $\mathrm{P_V}$ equation (1.5) with the indicated parameters.

The fifth Chazy equation is

$$\left(\frac{\mathrm{d}^2 v}{\mathrm{d}z^2} - 2v^3 - \alpha_1 v - \beta_1\right)^2 = 4\tan^2 z \left(v - \frac{\delta_1}{\sin z}\right)^2 \left\{\left(\frac{\mathrm{d}v}{\mathrm{d}z}\right)^2 - v^4 - \alpha_1 v^2 - 2\beta_1 v - \gamma_1\right\}. \tag{A.10}$$

Let q_1 and q_2 be any two distinct roots of the quartic equation,

$$q^4 + \alpha_1 q^2 + 2\beta_1 q + \gamma_1 = 0,$$

and let

$$r = q_1 + q_2, \qquad \alpha = (r + 2\epsilon_2\delta_1 - \epsilon_1)^2/8,$$
$$\beta = -(r - 2\epsilon_2\delta_1 - \epsilon_1)^2/8, \qquad \gamma = -(r^3 + 2\alpha_1 r - 4\beta_1)/(8r),$$
$$\delta = \{r^3 + 2(\alpha_1 + 2)r + 4\beta_1\}/(8r).$$

Then the solution of equation (A.10) is

$$\sin z = \epsilon_2 \frac{1+t}{1-t}, \qquad \cos z = \frac{2\mathrm{i}\sqrt{t}}{1-t}, \tag{A.11}$$

$$v(z) = \frac{\epsilon_1 t w'}{w} + \frac{(r + 2\epsilon_2\delta_1 - \epsilon_1)w}{2(t-1)} - \frac{(r - 2\epsilon_2\delta_1 - \epsilon_1)t}{2(t-1)w} - \frac{2\epsilon_2\delta_1(t+1)}{2(t-1)} - \frac{\epsilon_1}{2}, \tag{A.12}$$

where $w(t)$ satisfies the $\mathrm{P_{VI}}$ equation (1.6) with the indicated parameters. (If $r = 0$ then $\beta_1 = 0$ and the above expressions hold except that γ and $1/2 - \delta$ become the roots of the quadratic equation $4x^2 + 2\alpha_1 x + \gamma_1 = 0$.)

The multiplicity of values of q_1 and q_2 and the various $\pm$ signs imply a large number of transformations from $\mathrm{P_{VI}}$ to $\mathrm{P_{VI}}$. These include the sixteen Okamoto transformations (2.10), the four Schlesinger transformations (2.7), the four Fokas–Yortsos transformations (2.5) and the involutions (1.9), (1.13) and $(t, w) \to (1/t, 1/w)$.

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